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PRIMARY TEACHER EDUCATION (PrimTEd) PROJECT

GEOMETRY AND MEASUREMENT WORKING GROUP

DRAFT TEACHING UNIT- PROPERTIES

Preamble

The general aim of this teaching unit is to empower pre-service students by exposing them to geometry and measurement, and the relevant pedagogical content that would allow them to become skilful and competent mathematics teachers. The depth and scope of the content often go beyond what is required by prescribed school curricula for the Intermediate Phase learners, but should allow pre-service teachers to be well equipped, and approach the teaching of Geometry and Measurement with confidence. Pre-service teachers should essentially be prepared for Intermediate Phase teaching according to the requirements set out in MRTEQ (Minimum Requirements for Teacher Education Qualifications, 2019). “MRTEQ provides a basis for the construction of core curricula for Initial Teacher Education (ITE) as well as for Continuing Professional Development (CPD) Programmes that accredited institutions must use in order to develop programmes leading to teacher education qualifications.” [p6].

Target Audience

For utilisation by teacher-educators for the education of Intermediate Phase mathematics pre-service teachers.

“Big Ideas” in Geometry and Measurement

A “big idea” can be described as those core principles or theories that serve as a focal point for curricula. These “big ideas” reflect expert understanding, and while they do not necessarily need to originate as concepts or skills, they anchor discourse, inquiry, and argument in fields of study, incorporating the most meaningful content of study. (McTighe & Wiggins, 2004)

It is also argued that a focus on “big ideas” in teaching mathematics deepens teachers’ subject knowledge and can promote the development of relevant pedagogies, therefore emphasizing mathematical connections. (Barclay and Barnes, 2013)

Wide research supports this argument in many other fields, with researchers consistently finding that experts who operate off highly developed knowledge structures, which are more often than not organized around central concepts, or “big ideas”. (Niemi, et al. 2006)

This section will address *properties* as a constituent of the concept of geometry.

Rationale

The South African National Curriculum Statement makes reference to examples within the environment as a point of departure to infer geometric elements (objects and figures). (National Curriculum Statement, CAPS, FP Mathematics Gr R-3, pp21 & 22, and IP Mathematics Gr 4-6, pp21 & 22). By implication, it seems that the curriculum statement assumes or recommends that examples of geometric elements can be drawn from the learner’s surroundings, and that learners may be aware, prior to any school intervention, of the properties possessed by these elements as they recognise, compare and describe these elements. The awareness of similarities and differences may thus reside as innate or core knowledges within learners, even before the start schooling.

Izard, Pica, Spelke, and Dehaene conducted experiments with participants from an indigene group in the Amazon, the Mundurucu, as well as adults and age-matched children controls from the United States and France, and younger US children without education in geometry…

In short, their research tells us that: *"…at all ages, children and adults can use* ***distance*** *relationships."*

"Adults in both cultures also located a target by analysing two other fundamental properties of Euclidean geometry: ***angle*** *(the information that distinguishes corners of a triangle that differ in size) and* ***sense*** *(the information that distinguishes a form from its mirror image)"* (Ibid)

The above indicates that distance, angle and sense inform observation and perception of the observed or experienced environment, providing identification, description, and comparison of the world around us. This is in line with foundational guidelines from curricula, more specifically the South African schools' curriculum. (National Curriculum Statement, CAPS, FP Mathematics Gr R-3, pp21 & 22, and IP Mathematics Gr 4-6, pp21 & 22).

Properties of Geometric Elements

This will suggest that observed properties are a fundamental idea which underpins the teaching of geometry on schools. Taking into consideration "properties of geometric elements", let us first attempt to understand the two key words in this title, *geometry* and *properties.*

"The branch of mathematics concerned with the properties and relations of points, lines, surfaces, solids,..."

"A branch of mathematics that deals with the measurement, properties, and relationships of points, lines, angles, surfaces, and solids; broadly: the study of properties of given elements that remain invariant under specified transformations." (Merriam-Webster Dictionary)

A brief online search indicated that the word "property" generally makes reference to *ownership* or *possession.* If one only considers possible mathematical interpretations of the word "property", the following definitions become apparent:

"An attribute, quality, or characteristic of something." (Oxford Living Dictionaries)

"An essential or distinctive attribute or quality of a thing" (Dictionary.com)

Reference to *elements* is made previously, rather than only to objects and shapes, as points and lines must also be considered when teaching geometry. Hence the term *element* attempts to incorporate points, lines, shapes (figures), and objects.

In order to understand the importance of *properties* in a geometric context, we need to develop an understanding of the concept of "property".

In geometry, therefore, we may assume that the properties of geometric elements make these elements to be what they are. In other words, if any element should lose, or change any of its properties, it will not be what it is. It will not remain invariant. It will be transformed into another element.

This overview then suggests that one of the most important underpinning ideas in the teaching and learning of geometry in school mathematics is those *properties* which define geometric elements to be what they are.

The diagram which follows illustrates an envisaged structure which proposes the properties of geometric elements as being a point of departure to developing understanding in teaching and learning geometry.

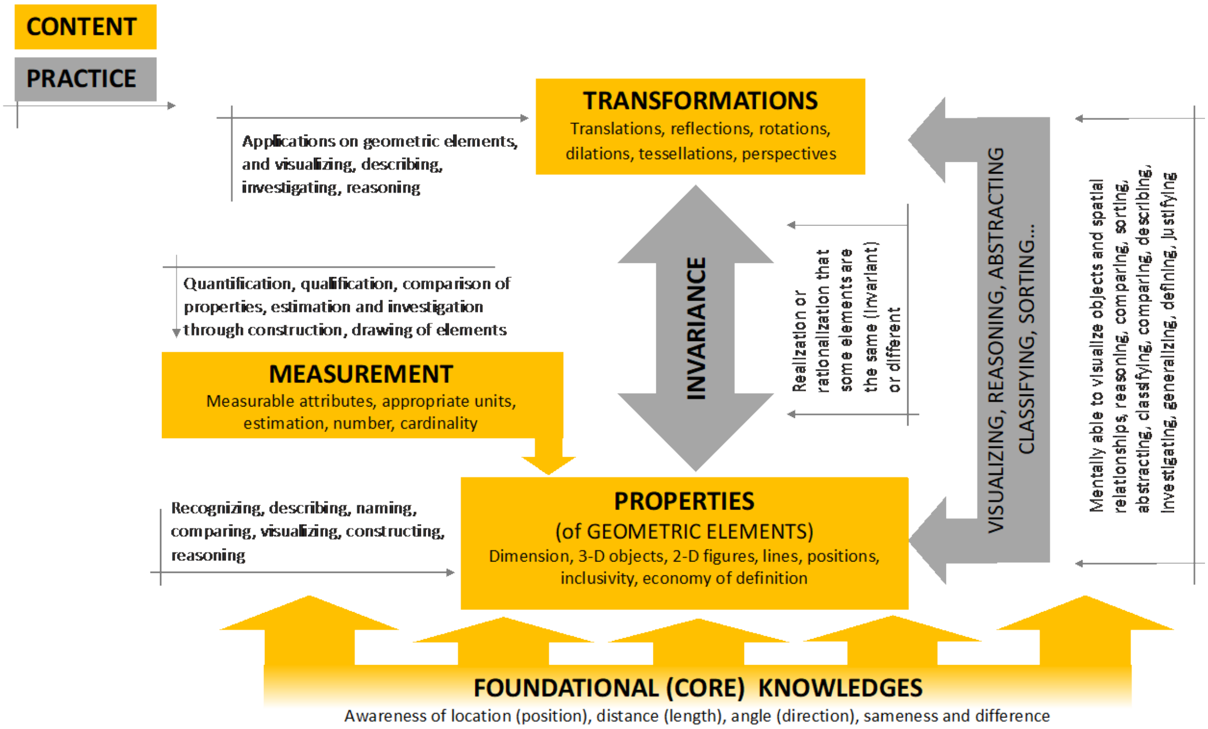

Diagram 1

In Diagram 1 above, the geometric elements, or rather the *properties* that give them existence, or that they give existence to, form the basis of our engagement with geometry. Without the geometric elements, based on these properties, there would be nothing to discuss, argue, or ponder.

The *properties* themselves can be described to various degrees of accuracy by measurement, whether by qualification or quantification, or even direct comparison. Drawing and construction also plays a role in formalizing the properties of geometric elements.

Measurement, therefore, plays an ongoing and significant role in the identification and definition of geometric points, lines, shapes and objects. (See Measurement Teaching Unit)

Once prototypes of geometric elements have been established, the ability to mentally *visualize* the same, similar, or different objects, and resulting spatial relationships can start to develop. This ability (to visualize) also continues to play an important role in the development of spatial reasoning.

Geometric elements, if subjected to rigid *transformations*, retain their properties. Of course, these geometric elements can lose some or all of their properties when subjected to transformations. For the purposes of school geometry, transformations generally maintain the properties of geometric elements. (See Transformations Teaching Unit)

Once geometric objects are engaged with, and rationalized, from the point of their properties (importantly), a realization of sameness and difference can be attained. In geometry, the importance of *invariance* becomes evident.

How does the South African schools curriculum present properties in geometry?

The South African school curriculum (CAPS) determines the content relating to properties of only *shapes and objects* in the Intermediate Phase. What follows are extracts from the curriculum document that determines the scope of what learners in the Intermediate Phase need to learn in relation to the properties of geometric elements (in this case, shapes and objects).

CAPS

SPECIFICATION OF CONTENT (PHASE OVERVIEW) SPACE AND SHAPE (GEOMETRY)			
• The main progression in Space and Shape (Geometry) is achieved by a focus on new properties and characteristics of 2-D shapes and 3-D objects in each grade. • Learners are given opportunities to identify and describe characteristics of 2-D shapes and 3-D objects and to develop their abilities to classify shapes and objects in the Senior Phase			
TOPICS	**GRADE 4**	**GRADE 5**	**GRADE 6**
3.1 **Properties of 2-D shapes**	**Range of shapes** • Recognize, visualize and name 2-D shapes in the environment and geometric settings - regular and irregular polygons – triangles, squares, rectangles, other quadrilaterals, pentagons, hexagons - circles **Characteristics of shapes** • Describe, sort and compare 2-D shapes in terms of - straight and curved sides - number of sides	**Range of shapes** • Recognize, visualize and name 2-D shapes in the environment and geometric setting, focusing on • regular and irregular polygons - triangles, squares, rectangles, other quadrilaterals, pentagons, hexagons, heptagons • circles • similarities and differences between squares and rectangles **Characteristics of shapes** • Describe, sort and compare 2-D shapes in terms of - straight and curved sides - number of sides - lengths of sides - angles in shapes, limited to ◊ right angles ◊ angles smaller than right angles ◊ angles greater than right angles **Further activities** • Draw 2-D shapes on grid paper	**Range of shapes** • Recognize, visualize and name 2-D shapes in the environment and geometric settings, focusing on - regular and irregular polygons - triangles, squares, rectangles, parallelograms, other quadrilaterals, pentagons, hexagons, heptagons, octagons - circles - similarities and differences between rectangles and parallelograms **Characteristics of shapes** • Describe, sort and compare 2-D shapes in terms of - number of sides - lengths of sides - sizes of angles ◊ acute ◊ right ◊ obtuse ◊ straight ◊ reflex ◊ revolution **Further activities** • Draw 2-D shapes on grid paper • Draw circles, patterns in circles and patterns with circles using a pair of pair of compassess

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CURRICULUM AND ASSESSMENT POLICY STATEMENT (CAPS)

TOPICS	GRADE 4	GRADE 5	GRADE 6
3.1 **Properties of 2-D shapes**	**Further activities** • Draw 2-D shapes on grid paper	**Angles** • Recognize and describe angles in 2-D shapes: - right angles - angles smaller than right angles - angles greater than right angles	**Angles** • Recognize and name the following angles in 2-D shapes: - acute - right - obtuse - straight - reflex - revolution
3.2 **Properties of 3-D objects**	**Range of objects** • Recognize, visualize and name 3-D objects in the environment and geometric settings, focusing on: - rectangular prisms, - spheres - cylinders - pyramids **characteristics of objects** • Describe, sort and compare 3-D objects in terms of - shapes of faces - flat and curved surfaces **Further activities** • Make 3-D models using cut out polygons	**Range of objects** • Recognize, visualize and name 3-D objects in the environment and geometric settings, focusing on: - rectangular prisms and other prisms - cubes - cylinders - cones - pyramids - similarities and differences between cubes and rectangular prisms **characteristics of objects** • Describe, sort and compare 3-D objects in terms of - shape of faces - number of faces - flat and curved surfaces **Further activities** • Make 3-D models using cut out polygons • Cut open boxes to trace and describe their nets	**Range of objects** • Recognize, visualize and name 3-D objects in the environment and geometric settings, focusing on - rectangular prisms - cubes - tetrahedrons - pyramids - similarities and differences between tetrahedrons and other pyramids **characteristics of objects** • Describe, sort and compare 3-D objects in terms of - number and shape of faces - number of vertices - number of edges **Further activities** • Make 3-D models using: - drinking straws, toothpicks etc - nets

Curriculum and Assessment Policy Statement (CAPS) Intermediate Phase Grades 4-6 Mathematics (2011)

The curriculum document attempts to develop an awareness of properties through the recognition and description of geometric elements, namely 2-D shapes (figures) and 3-D objects.

It goes on to suggest the visualization, comparison and sorting of geometric elements (in this case only shapes and objects). Again, these activities seek to develop an awareness and knowledge of the properties of these geometric elements.

The intent is apparently to develop a manner of thinking about the properties of shapes and objects, and through comparisons, looking at similarities and differences, arrive at authentic "definitions", thus paving the way to thinking about, and recognizing shapes and objects *on the basis of their properties.*

In Grade 4, the curriculum requires learners to be able to "recognise, visualize and name" 2-D shapes, namely triangles, squares, rectangles, "other quadrilaterals", pentagons and hexagons, as well as circles. They must also "recognise, visualize and name" 3-D objects, specifically rectangular prisms, cylinders, spheres, and pyramids. Learners are also expected to be able to "describe, sort and compare" shapes in terms of sides, whether these are straight of curved, and also to count the number of sides that shapes possess. Similarly, they need to "describe, sort and compare" 3-D objects in terms of flat and curved surfaces and the shapes of faces. The requirement is also made for learners to draw shapes and build objects – presumably to explore properties through "construction".

In Grade 5, the same skills/activities are required, but the range of shapes in increased to include heptagons in the case of 2-D figures (shapes) and including cones in the case of 3-D objects. Learners are also required to be able to recognise and describe the similarities and differences between squares and rectangles in 2-D, and the similarities and differences between cubes and rectangular prisms in 3-D (no mention is made of square prisms). When considering 3-D objects, learners also have to be able to consider the number of faces on any polyhedron.

Angles are introduced in Grade 5, with minimum requirements to recognise angles less than, and greater than ninety degrees (acute and obtuse angles). This is alarming, as the concept of angles has not been covered in previous grades.

"A plane angle is the inclination to one another of two lines in a plane, which meet each other, but do not lie in a straight line." - Euclid, *Elements*, Definition 8 [A: Euclid]

The space (usually measured in degrees) between two intersecting lines or surfaces at or close to the point where they meet. - Oxford Living Dictionaries, https://en.oxforddictionaries.com

A broader, all-encompassing description of angles in the school mathematics context is noted by Henderson and Taimina, and later by Fyhn, who assert that angles can be categorized and defined using three perspectives, namely: *geometric shape*; *dynamic movement*; and *measure* (Fyhn, 2007; Henderson & Taimina, 2005).

In addition to constructing 2-D figures and 3-D objects, learners are also required to represent the nets of geometric objects (polyhedrons).

In Grade 6, the same skills/activities are required, but the range of shapes is increased to include octagons and parallelograms. Learners are also required to be able to recognise and describe the similarities and differences between rectangles and parallelograms in 2-D, and tetrahedrons and other pyramids in 3-D. Cognizance it taken of the number of faces, edges and vertices, as well as the shapes of faces.

It is within the context of the curriculum, but guided by mathematics content knowledge, that the following content standards are written.

Content Standards for Properties

1. Knowledge of Geometrical Properties
   - In Geometry, we may assume that the properties of geometric elements make these elements to be what they are. In other words, if any element should lose, or change any of its properties, it will no longer be what it was. It will not remain invariant; it will be transformed.
   - This suggests that one of the most important ideas underpinning the teaching and learning of geometry in school mathematics is those *properties* which define geometric elements to be what they are.
   - Further, properties are most likely to be those attributes which are initially perceived when any geometric element is observed.

1.1 Awareness of core knowledges which learners possess prior to the commencement of formal schooling.

a) Ability to describe position relative to other positions or markers
b) Estimation and comparison of distances and lengths
c) Ability to indicate direction or describe an angle in terms of directions
d) Awareness of sameness and difference and similarities
e) Recognition of invariance after transformations

1.2 Understanding dimension

To understand how lived space and objects can be described through reference to a minimum number of points in space.

a) Understand that dimension is a mathematical construct which may be applied to everyday contexts

b) To understand that that any one dimension is direction along any one straight path, and that subsequent dimensions are defined according to previously established directions in a consistent way [how to bring in polar groups]

c) Can exxplain 0-D (point), 1-D (line), 2-D (shape), and 3-D (object)

1.3 Classifying 3-dimensional objects according to properties

a) Using at least one criterion to classify/group objects – with reasons
b) Justifying the sorting of geometric objects
c) Identifying geometric objects according to their properties
d) Describing geometric objects according to their properties

1.4 Rational classification of 2-dimensional shapes according to observed properties

a) Using at least one criterion to classify/group shapes – with reasons
b) Justifying the sorting of geometric shapes
c) Identifying geometric shapes according to their properties
d) Describing geometric shapes according to their properties

1.5 Realisation of inclusivity with regard to objects and shapes

a) Understanding that, for example, all squares are rectangles, but that not all rectangles are squares
b) Understanding that, for example, all cubes are also square prisms

1.6 Economy of definitions

a) Ability to interpret and describe objects and shapes in terms of minimal properties

Theories, Teaching Approaches and Methodology

Both inductive and deductive approaches will be modelled with this teaching unit. Teaching methodology will include inquiry/investigation, and discussion, hypothesizing, and modelling.

Sarama and Clements and Battista

However, building intuition based on experience cuts both ways. The limitations of human experience account not only for the adaptive and organizing functions of intuitions, but also for distorted or erroneous representations of reality. Thus, space intuitions, like other intuitions, do not develop inevitably into increasing correspondence with "pure" logic or mathematics, as a reading of Piaget may suggest. (Clements & Battista, 1992)

For students to function adequately at one of the advanced levels in the van Hiele hierarchy, they must have mastered large portions of the lower levels {Hoffer, 1981 #44}. Progress from one level to the next is more dependent upon instruction than on age or biological maturation. Teachers can "reduce" subject matter to a lower level, leading to rote memorization, but students cannot bypass levels and achieve understanding (memorization is not an important feature of any level). The latter requires working through certain "phases" of instruction. (Clements & Battista, 1992)

Concepts implicitly understood at one level become explicitly understood at the next level. "At each level there appears in an extrinsic way that which was intrinsic at the preceding level. At the base level, figures were in fact also determined by their properties, but someone thinking that this level is not aware of these properties" {van Hiele, 1984 #97, p. 246} (Clements & Battista, 1992)

The paucity of geometry before high school is a major concern. The usual preschool to middle school curriculum includes little more than recognizing and naming geometric shapes (Porter, 1989). Through the grades, the curriculum tends to name more geometric objects but not require deeper levels of analysis (Fuys, Geddes, & Tischler, 1988). Compounding matters, teachers often do not teach even the barren geometry curriculum that is available to them. Fourth- and fifth-grade teachers across entire districts spend "virtually no time teaching geometry" (Porter, 1989, p. 11): Current practices in the primary grades also promote little conceptual change: First-grade students in one study were more likely than older children to differentiate one polygon from another by counting sides or vertices (Lehrer, Jenkins, & Osana).

Suggested sequence of conceptual development activities

Unit	Topic	Focus	Content Standard
1	Dimension	The properties of 0-; 1-; 2-; and 3-D. What makes geometric elements 3-dimensional or 2-dimensional, or even 1- or 0-dimensional?	1.1 and 1.2
2	3-D objects	3-D objects are what they are in that they possess three dimensions, namely length, breadth and height. The "exit" point from this activity arrives at flat-faced solids	1.3
3	Polyhedrons	Flat-faced solids (polyhedrons) are of particular interest in Euclidean Geometry – especially since they are ideal in terms of their properties and our ability to define them. The focus remains on classification based on properties	1.3
4	Prisms and pyramids	Among polyhedrons, pyramids and prisms are especially interesting, and certainly definable – again the focus remains on classification based on properties	1.3
5	2-D shapes	Shapes are accessed through regard of faces as being 2-D. This can be introduced through nets of polyhedrons. Various shapes are explored, including open and closed shapes, shapes which have curved sides only, both curved sides and straight sides, and shapes with only straight sides. Focus remains on classification based on properties.	1.4
6	Polygons	Shapes which are closed and have only straight sides are of particular interest. These, within the context of school geometry, possess obvious properties and are easily definable. Focus remains on classification based on properties	1.4
7	Quadrilaterals	Among polygons, quadrilaterals are especially interesting, and certainly definable – again the focus remains on classification based on properties. Squares are rectangles are parallelograms are trapeziums are quads… Squares are rhombi are kites are quads…	1.5
8	Economy of definition	Using minimal properties to accurately define geometric objects and shapes.	1.6

Conceptual Development Activities

Unit 1: Understanding Dimension

Content standard which this activity addresses

Awareness of core knowledges which learners possess prior to the commencement of formal schooling.

a) Ability to describe position relative to other positions or markers
b) Estimation and comparison of distances and lengths
c) Ability to indicate direction or describe an angle in terms of directions
d) Awareness of sameness and difference and similarities
e) Recognition of invariance after transformations

Understanding dimension

a) An awareness of 0-D (point), 1-D (line), 2-D (shape), and 3-D (object)
b) To understand that any one dimension is a direction along any one straight path, and that subsequent dimensions are perpendicular to previously established directions
c) Understand that dimension is a mathematical construct which may be applied to everyday contexts

Intent of this activity

It would first be necessary to clarify the content area title as specified by the SA school's curriculum, namely *Space and Shape*. The first task would be to set the students thinking about what this title could mean, and why/if the two geometric elements mentioned in the title are ordered in a particular manner. Initially, we would want students to think about the terms *space* and *shape*, and pen down their own definitions of these two elements.

Habits of mind which are to be developed

- Visualising
- Inquiring
- Observing
- Describing
- Tenacity
- Justifying
- Reasoning

Questions and explanations

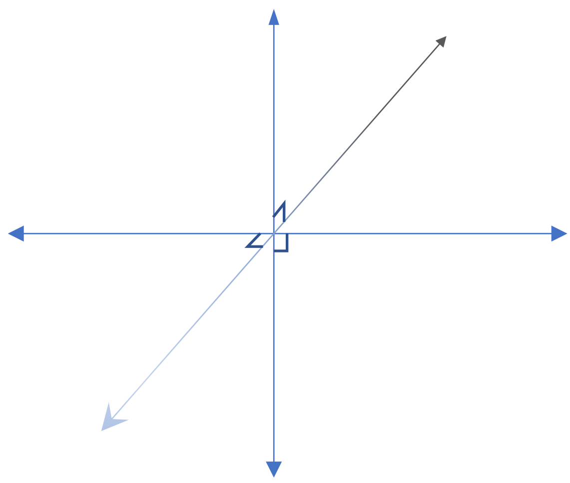
What is space and what is shape?

Space would be a 3-dimensional conception, while shape references a 2-dimensional plane. We could ask if *shape* could exist outside of *space*.

In terms of the use of natural numbers (positive whole numbers) in a nominal manner, and as dimension (within 2- and 3-D) can be counted, it makes sense that dimensions exist as aligned to numbers. Therefore, counting backwards, 3-D; 2-D… we should then get 1-D and 0-D. Perhaps we could count on from 3-D and get 4-D; 5-D; 6-D; etc. as well!

What, then, is a "dimension"?

A dimension may be described a direction along a straight path. For the purpose of "order", any subsequent dimension cannot be haphazardly oriented in relation to the first dimension. Any subsequent dimension is then perpendicular to the first. A third dimension would then be oriented perpendicularly to the first and second dimensions.



So, if 2-D and 3-D exist, it makes sense that there should be a 1-D… and possibly a 0-D as well!

Let's explore this…

Students will be asked to draw four columns, and then make a list of incidences of 3-D examples in the room, followed by 2-D, then 1-D, and finally 0-D. Once each "dimension" is listed, the lecturer should discuss why each of these are 3-dimensional

Typically, they would list as follows (with some guidance):

3-D	2-D	1-D	0-D
Chair Table Bag Chalk board Computer Pencil bag Person	Shadow Images in a photo Reflection Image on screen Writing on board?	The following regarded in a 1-D sense (as a straight line): Straight edge of the table Corner of the room where two walls meet Line of sight?	The following regarded in a 0-D sense (as a location or point): Corner of the table/door The corner of the room where the two walls meet the ceiling

The lecturer, with input from the students, would point out the three different dimensions, and why each of the mentioned objects are 3-dimensional. The intent is to focus on those properties which make 3-D objects what they are. These objects will have length breadth and height.

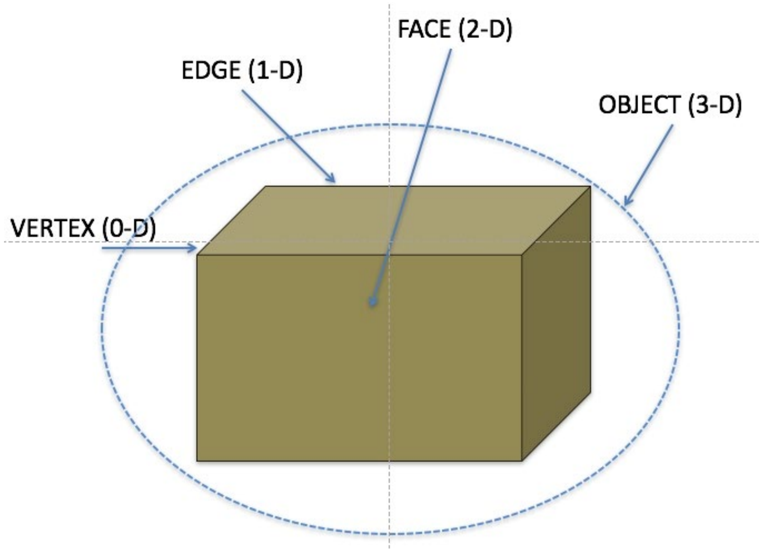
Similarly, the examples of 2-D are dealt with, showing where/how these two dimensions lie (perpendicular to each other). These objects will have length and breadth, or length and height, or breadth and height.

1-D will be an imaginary line, or referenced along a line of sight, or imagined along the edge of a table, chalkboard. It will only have length or breadth or height.

0-D is also dealt with, except that 0-D can only be a location, using a point of reference, or as an imaginary location having no length, no breadth, nor height.

The following activity relates terminology used to describe aspects of polyhedrons (flat-aced solids) in terms of 0-D; 1-D; 2-D and 3-D.

This activity shows how geometric terminology, used to name aspects of a cuboid (in this case, a cardboard box) relates to dimension. Most importantly, it conveys that it is a "way of seeing" these parts of the box in terms of being 0-D to 3-D. For example, the side of the box is really made of cardboard of a definite thickness, but we attempt to see it as a 2-D shape when we only consider it in terms of length and breadth.



Resources required

Reference to natural and manmade elements in the environment inside or around the classroom. Skewers to show the relationship between the three dimensions. You will need a box to show the incidences of dimension as perceived by a "mathematical eye".

Supporting and explanatory diagrams/videos

All explanatory diagrams are incorporated into the explanations above.

Research articles for support

References for further reading

Unit 2: 3-D OBJECTS

Content standards which this activity addresses

Rational classification of 3-dimensional objects according to observed properties

a) Using at least one criterion to classify/group objects – with reasons
b) Justifying the sorting of geometric objects
c) Identifying geometric objects according to their properties
d) Describing geometric objects according to their properties

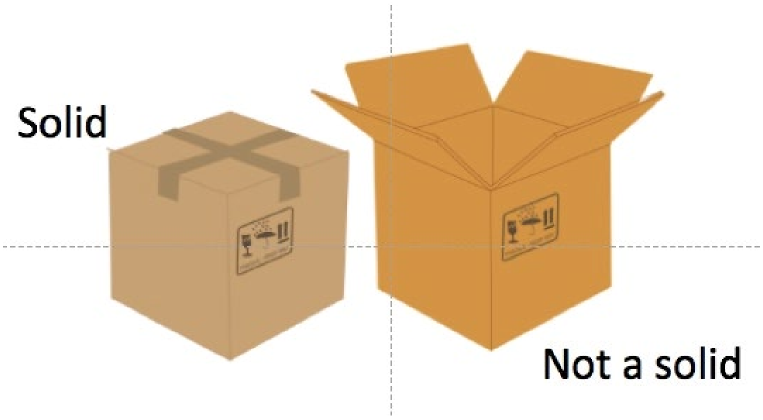
Intent of this activity

This activity will set out to get students to classify 3-D objects strictly according to their properties. While students can be allowed to explore the objects, and classify these according to their own criteria, a more "scientific" approach to classifying these objects needs to be encouraged by the facilitator. This may involve getting students to focus on one criterion at a time while they are sorting.

Of course, this realization must be arrived at through reasoning, and NOT through just naming objects. For this reason, a variety of "not so typical" objects must be presented to the students for them to sort. This will include spheres, ovoid objects, cylinders, cubes, cones, rectangular prisms, a variety of pyramids.

Students need to be able to know what solids are. This is necessary, as they will be working with geometric solids.

One of the initial conceptions that students will need to consider, as they classify objects, is whether these are solids or non-solids. When 3-D objects are closed, we call them SOLIDS Even if they are hollow inside, they are still deemed to be solids if they're closed all round.



Habits of mind which are to be developed

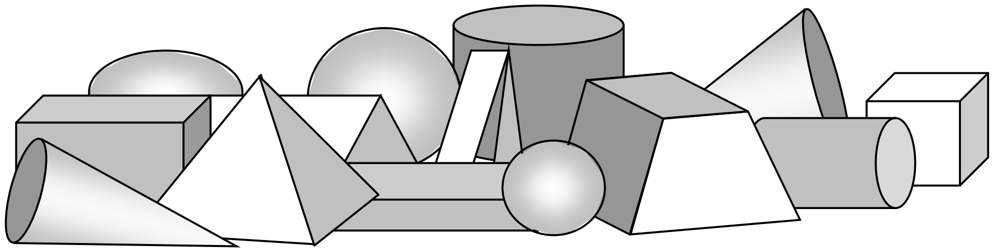
- Visualising
- Observing
- Inquiring
- Sorting
- Classifying
- Investigating
- Reasoning
- Comparing
- Justifying

Questions that can be asked

Students need to motivate groupings of objects by giving explanations as to why they have grouped these together.

Students should be asked to research the Van Hiele Levels and say which level they think that they are operating at in terms of spatial reasoning. It is recommended that students be provided with an accessible text which outlines the Van Hiele Model. In mathematics education, the Van Hiele Model is a theory that describes how students learn geometry. The theory originated in 1957 in the doctoral dissertations of Dina van Hiele-Geldof and Pierre van Hiele (wife and husband) at Utrecht University, in the Netherlands.

Hand out a variety of geometric objects to the students. These should include spheres, hemispheres, cylinders, cones, prisms (cubes, rectangular, square, triangular, hexagonal, pentagonal, octagonal`), pyramids (square-based, triangle-based, hexagon-based, etc).

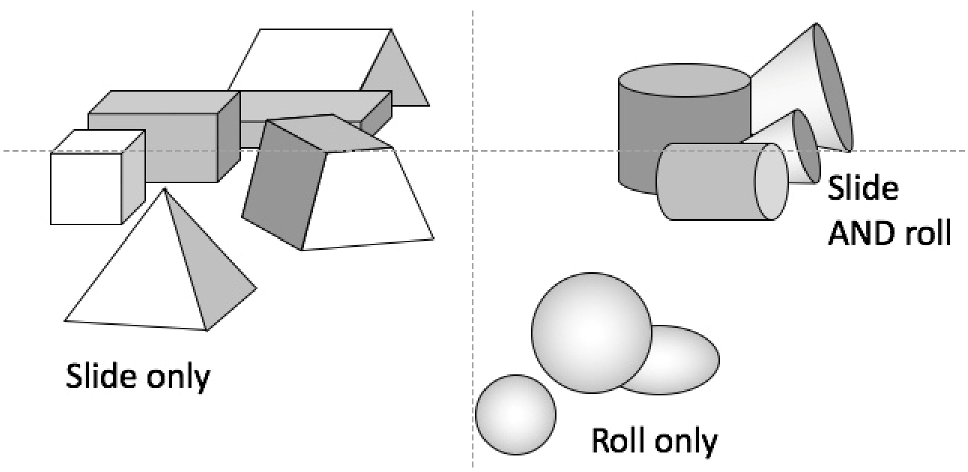


Students will be asked to sort these objects into THREE groups. Give reasons why they have grouped them in this way.

Of course, you will notice that students will tend to sort the objects into "those that look like triangles" or "those that are like circles" or "those that are like blocks". You may point out to them here that they may be operating at Level 0 of the Van Hiele Model!

The three potential groups that the students sort the objects into are:

- Those with only curved surfaces (roll only)
- Those with only flat surfaces (slide only)
- Those with curved and flat surfaces (roll and slide)



Most importantly…
WHY can some objects only slide?
WHY can some object only roll?
How come some objects can roll and slide?

NB: Sliding and rolling are not mathematical properties. The intent here is to arrive at mathematical properties…

Supporting and/or explanatory diagrams and videos

https://youtu.be/wM1KbsyojQY

Resources

Wooden blocks, assorted geometric solids, balls (for spheres), cylinders, ovoid forms, prisms, pyramids, etc.

Research articles for support

References for further reading

Unit 3: POLYHEDRONS

Content standard which this activity addresses

Rational classification of 3-dimensional objects according to observed properties

a) Using at least one criterion to classify/group objects – with reasons
b) Justifying the sorting of geometric objects
c) Identifying geometric objects according to their properties
d) Describing geometric objects according to their properties

Intent of this activity

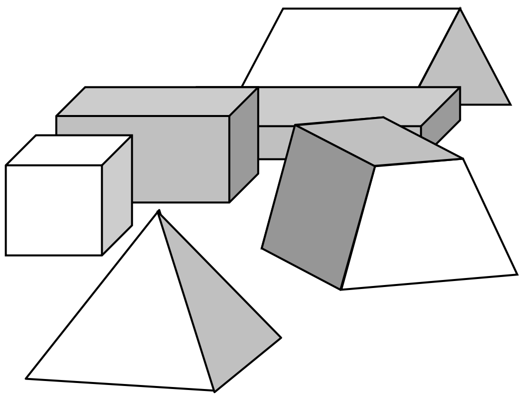
This activity sets out to get learners to describe, classify and eventually name polyhedrons according to the number of faces that they possess. It addresses conceptions as to which polyhedrons are which, previously identifiable and named purely on the bases of prototypes. So now, for example, what was previously known only as a "cube", can also be classified as a hexahedron (a polyhedron with six faces). Similarly, a square-based pyramid is also a pentahedron. Thus, the idea of classifying and naming is done according to observed properties rather than previous facsimiles.

Habits of mind to be developed

- Visualising
- Observing
- Inquiring
- Sorting
- Classifying
- Investigating
- Reasoning
- Comparing
- Justifying

Questions and explanations

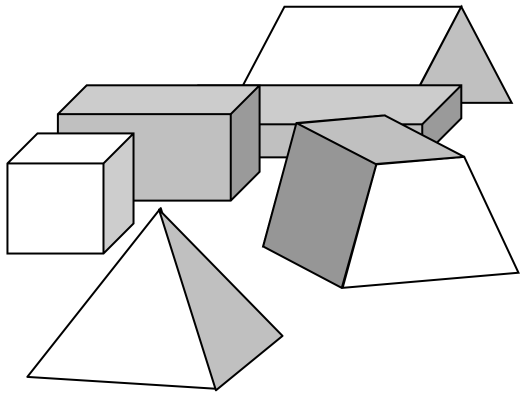
Let's take a closer look at those objects that can only slide...



These objects are solids, and they can only slide because they have flat faces
These are flat-faced solids
In Mathematics, flat-faced solids are called polyhedrons (or polyhedra)
"poly" means *many*, and "hedron" means *faces*
Polyhedrons are therefore, 3-dimensional flat-faced solids

It is very important that the focus remains on the properties, and that all objects have the same property/properties.

Now arrange these polyhedrons into groups according to the numbers of faces that they have



Were there any polyhedrons with fewer than four faces?

These groups can be named hence:
4 faces – tetrahedrons
5 faces – pentahedrons
6 faces – hexahedrons
7 faces – heptahedrons (septahedrons)
8 faces – octahedrons
9 faces – nonahedrons
10 faces – decahedrons
12 faces – dodecahedrons

Learners will realize that they can now classify and name polyhedrons. This is non-reliant on prototypes.

Resources required

Wooden blocks, assorted geometric solids, such as cubes, pyramids, prisms

Supporting and/or explanatory diagrams and videos

https://youtu.be/wM1KbsyojQY

Research articles for support

UNIT 4: PRISMS AND PYRAMIDS

Content standard which this activity addresses

Rational classification of 3-dimensional objects according to observed properties

a) Using at least one criterion to classify/group objects – with reasons
b) Justifying the sorting of geometric objects
c) Identifying geometric objects according to their properties
d) Describing geometric objects according to their properties

Intent of this activity

This activity provides learners with the opportunity to deduce prisms and pyramids while using provided properties of these two geometric objects. Learners are encouraged to explore geometric objects and classify them accordingly.

Habits of mind to be developed

- Visualising
- Observing
- Inquiring
- Sorting
- Classifying
- Investigating
- Reasoning
- Comparing
- Justifying

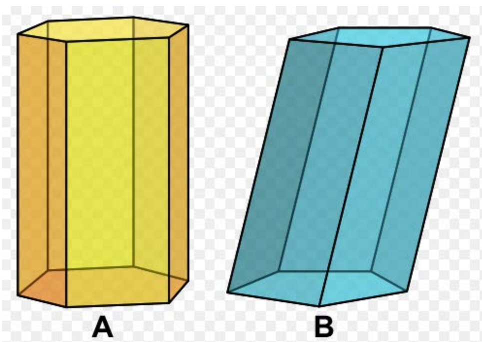
Questions and explanations

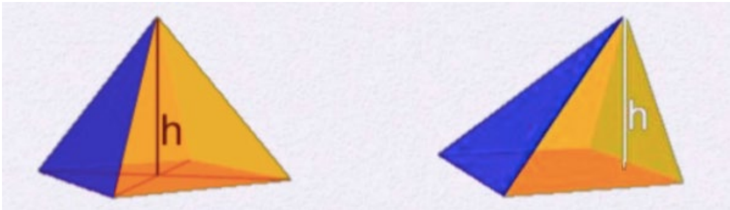
Polyhedrons can also be divided into two other groups, namely pyramids and prisms
With this activity, the properties of prisms and pyramids made be provided to students, and they will be asked to sort polyhedrons according to these properties.

Prisms	Pyramids
Have two bases Named after their bases Lateral faces are vertically parallel Lateral faces are squares/rectangles/rhombi/parms Bases are same size and shape (congruent) Bases are parallel to each other	Have one base Named after their base Lateral faces converge at an apex Lateral faces are triangles

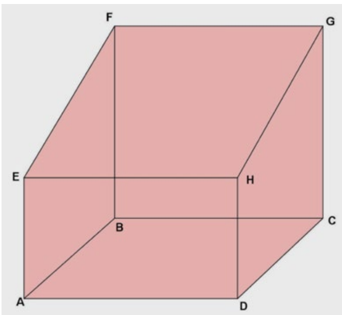
Students are also introduced to the concepts of right and oblique. This can be incorporated into the naming of pyramids and prisms.

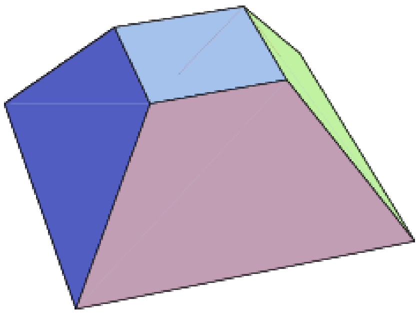
Prisms and pyramids can be *right* or *oblique*:





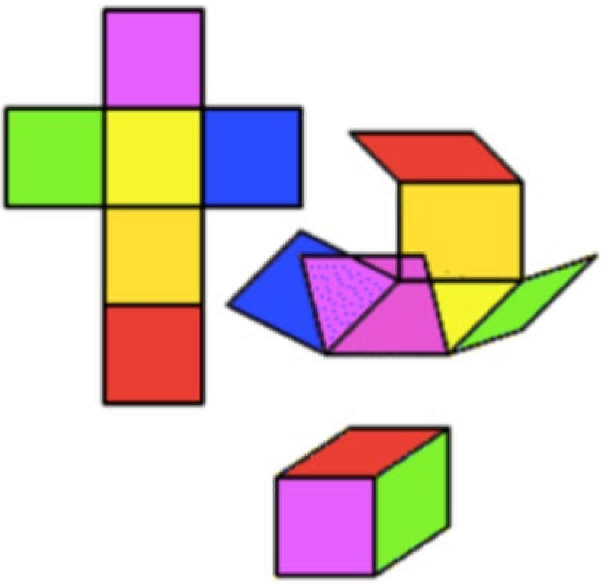
Prisms and pyramids can be also be truncated:
Students are introduced to the idea of truncation. This can be incorporated into the naming of pyramids and prisms.





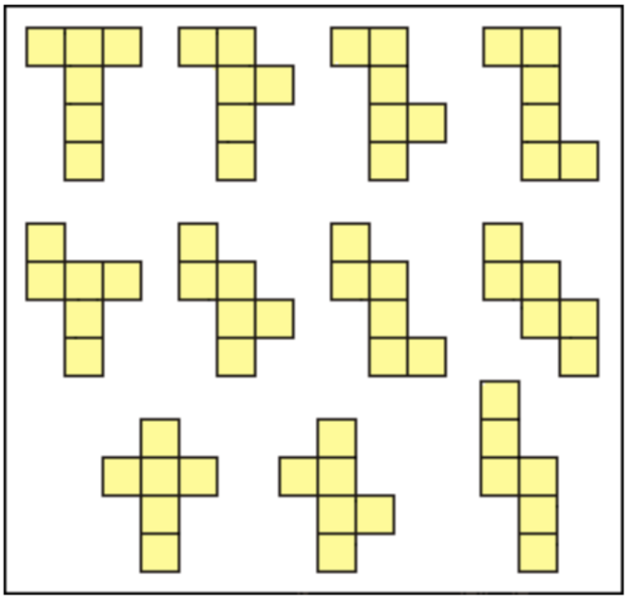
You will have noticed that it is very hard to speak about polyhedrons, without mentioning shapes. This is because these shapes make up these polyhedrons.
So, let’s look at shapes.

We can imagine any polyhedron as a net…



See how a cube (3-D) can be expressed as a 2-D net

How many nets of a cube can you make?



Resources required

Wooden blocks, assorted geometric solids, such as cubes, pyramids, prisms

Supporting and/or explanatory diagrams and videos

https://www.youtube.com/watch?v=Dz6nFwgKIZ4

Research articles for support

References for further reading

UNIT 5: CLASSIFYING 2-D SHAPES

Content standard which this activity addresses

Rational classification of 2-dimensional shapes according to observed properties

a) Using at least one criterion to classify/group shapes – with reasons
b) Justifying the sorting of geometric shapes
c) Identifying geometric shapes according to their properties
d) Describing geometric shapes according to their properties

Intent of this activity

Having investigated the nets of various polyhedrons, learners would have understood that these, polyhedrons, comprise of 2-dimensional shapes. Each flat surface comprises a straight-sided, closed shape.

This activity will set out to get learners to realise polygons as closed, 2-dimensional, straight-sided figures, through a process of sorting.

The facilitator of this activity may wish to allow the learners to sort the provided shapes – with reasons given for every grouping, without providing any guidelines in terms of the properties – allowing learners to "discover" the polygons. Of course, the facilitator will wish to reach a grouping of closed, 2-dimensional, straight-sided figures. While there may be one or two complex polygons among the shapes to be sorted, these should present a challenge in terms of later classification according to the number of corners/sides. The facilitator can explain that complex polygons are conceptions which exist but will be dealt with during further investigations in mathematics.

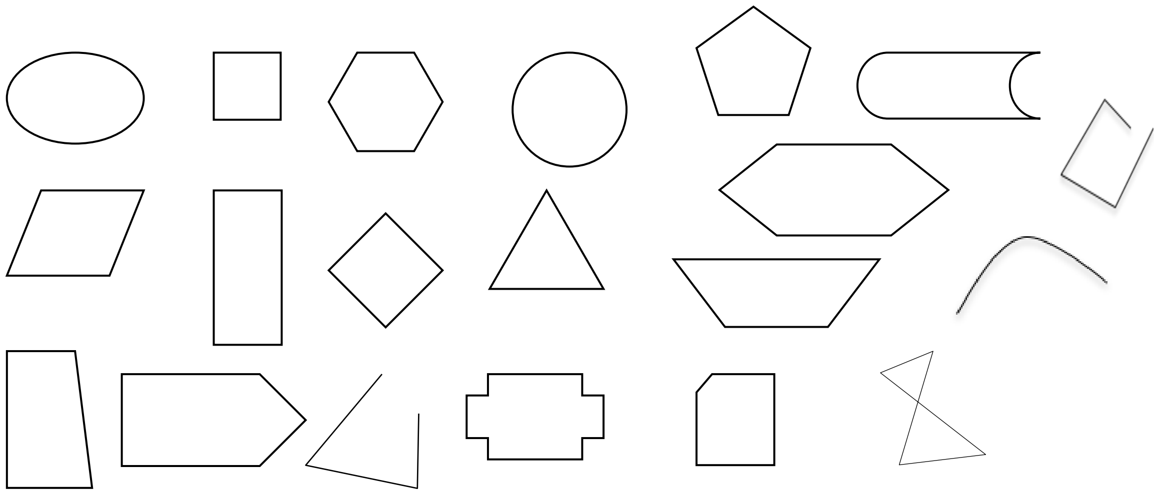
Habits of mind to be developed

- Visualising
- Observing
- Inquiring
- Sorting
- Classifying
- Investigating
- Reasoning
- Comparing
- Justifying

Questions and explanations

Hand out a variety of shapes to your students. These should be separate, and printed onto pages so that students can sort, arrange, re-arrange, as they discuss groupings.

To streamline this activity, you can ask learners to remove all shapes that are open, then remove shapes that have curved sides (sides that are not straight).



Using the shapes that you've been given, sort them accordingly:
Remove all the shapes that are open
Now you have only a group of closed shapes

Remove all the shapes that have curved or "wavy" sides
Now you have a group of closed shapes with only straight sides

Resources required

Assorted shapes, both open and closed, with and without straight sides, printed onto paper (not cut out).

Supporting and/or explanatory diagrams and videos

https://www.youtube.com/watch?v=l4S2dX3OLL0

Research articles for support

References for further reading

UNIT 6: NAMING POLYGONS!

Content standard which this activity addresses

Rational classification of 2-dimensional shapes according to observed properties

a) Using at least one criterion to classify/group shapes – with reasons
b) Justifying the sorting of geometric shapes
c) Identifying geometric shapes according to their properties
d) Describing geometric shapes according to their properties

Intent of this activity

Having identified polygons as closed, 2-D, straight-sided shapes, learners will now have to be able to further classify polygons into sub-groups, according to the number of corners/sides. At this point, the approach of using reasoning, rather than depending purely on recognition of prototypes (Level 0 of the van Hiele Levels) becomes very important. In most classrooms, learners would have engaged with only regular polygons, along with rectangles as "shapes" – as is evident on so many classroom education posters. By using a keen focus on reasoning through observing the properties possessed by polygons, learners will discover that some of the most "non-typical" shapes are also pentagons, hexagons, octagons. They will also realise that a square oriented without vertical and horizontal sides is still a square. Finally, they will learn that each group has its own name. Finally, learners will realise that isosceles triangles, rectangles and rhombi are not regular polygons.

Habits of mind to be developed

- Visualising
- Observing
- Inquiring
- Sorting
- Classifying
- Investigating
- Reasoning
- Comparing
- Justifying

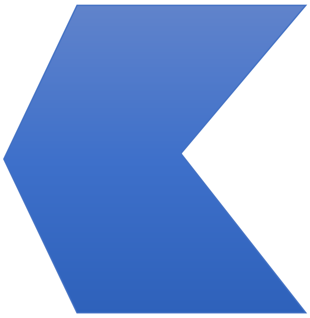
Questions and explanations

Polygons are 2-dimensional shapes which are closed, have straight sides only

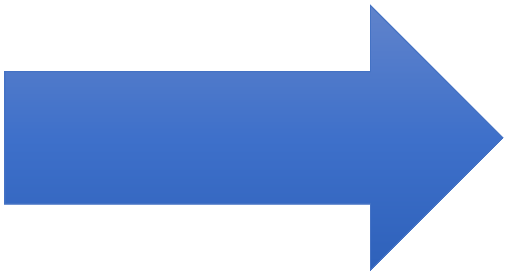
What's the difference between a polygon and a polyhedron?
Now group your polygons according to the number of sides…

Name these groups accordingly:
3 sides – triangles
4 sides – quadrilaterals
5 sides – pentagons
6 sides – hexagons
7 sides – heptagons (septagons)
8 sides – octagons
9 sides – nonagons
10 sides – decagons
11 sides – 11-gons

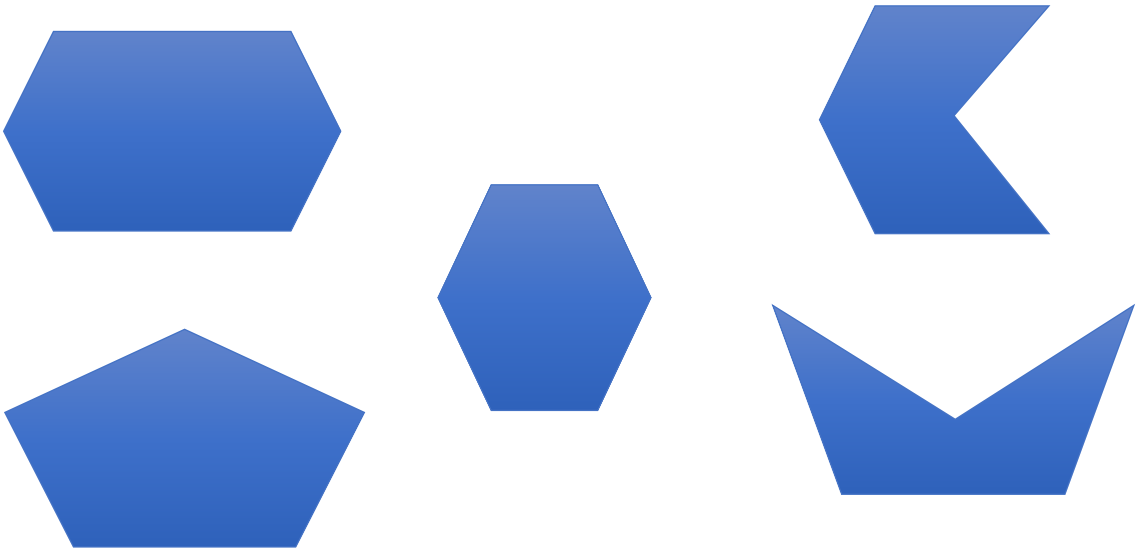
Did you ever imagine that you would name these:
Learners would mostly only have encountered geometric shapes which are “typical” (rectangles, parallelograms, rhombi, etc.) or regular polygons.


a hexagon!

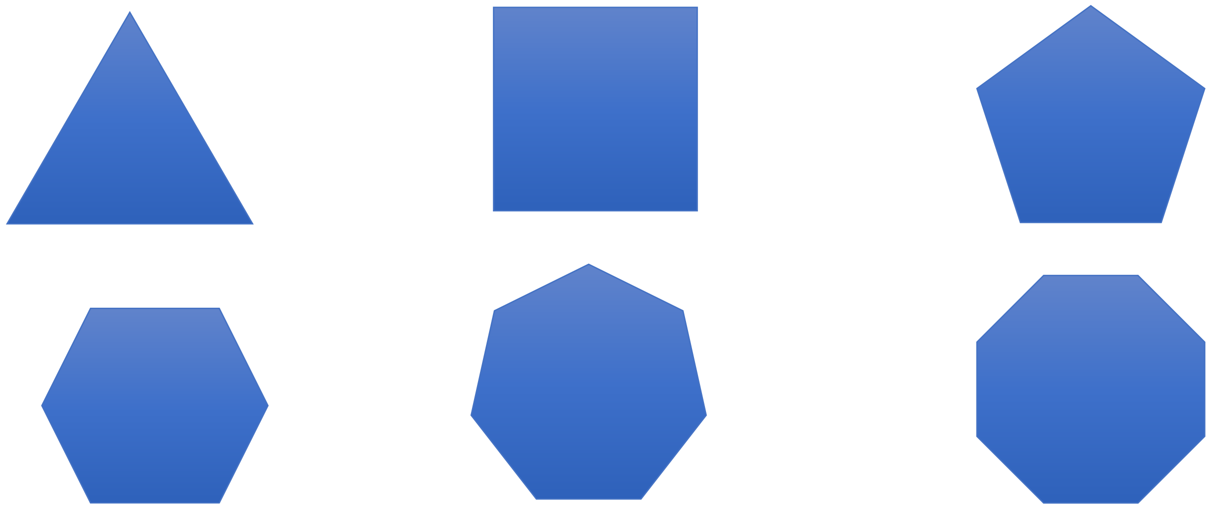

a pentagon!


a heptagon!

Some polygons are convex, and some are concave:
Here you will show another broad classification of polygons, namely those which are convex and those which are concave (concave polygons have at least one interior angle which is >180°)

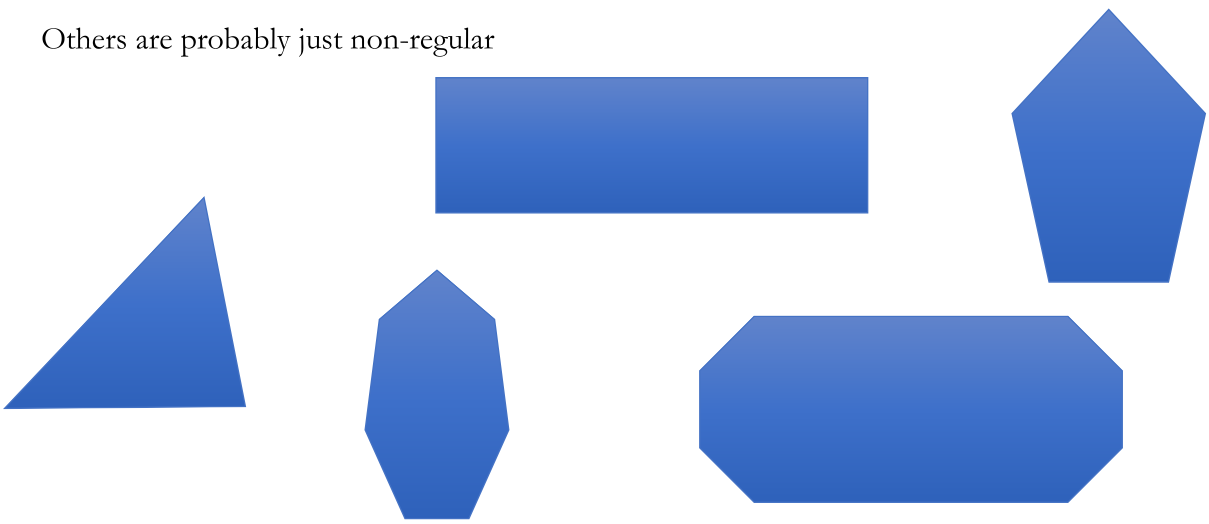


Some polygons are equilateral AND equiangular
Students are introduced to the concepts of regular and non-regular. This can be done by construction, by investigation, or maybe the properties of regular polygons are provided, and learners must construct, or identify these.



These are REGULAR POLYGONS

Others are probably just non-regular



REMEMBER: We name shapes according to their properties…

Quadrilaterals provide an interesting point of discussion for learners, especially when regarded in terms of their properties. Note that the names of the quads here are not initially important. The focus MUST be maintained on the properties. You may have to stop and discuss some or all of these quadrilaterals in terms of their properties. You may find that some learner may come up with properties which may be superfluous (a rectangle has two long sides and two short sides). It is essential that you stick to the correct properties (and maybe definitions) of these polygons (quadrilaterals).

What follows is a series of diagrams that illustrate that squares, for example, possess the properties of rectangles, and therefore we can say that all squares are rectangles.

Resources required

Polygons, as printed on paper, having been sorted during the previous activity.

Supporting and/or explanatory diagrams and videos

https://www.youtube.com/watch?v=l4S2dX3OLL0

Research articles for support

References for further reading

UNIT 7: QUADRILATERALS (QUADS)

Content standard which this activity addresses

Realisation of inclusivity with regard to objects and shapes

a) Understanding that, for example, all squares are rectangles, but that not all rectangles are squares
b) Understanding that, for example, all cubes are also square prisms

Intent of this activity

Having named polygons purely according to their properties, and "breaking away" from examples based on facsimiles ("this shape is a square, that shape looks like this shape, so therefore that shape is also a square"), this activity sets about looking at properties, and seeing which essential properties are possessed by otherwise defined and named polygons. Quadrilaterals present an excellent opportunity to identify and compare properties in terms of polygons. Perhaps it is because that having four sides, they have distinctly opposite sides and distinctly opposite corners/angles. They also have distinctly adjacent sides and distinctly adjacent corners/angles. This activity will then demonstrate that squares possess all those necessary properties of rectangles, as well as those of rhombi, kites, and trapeziums.

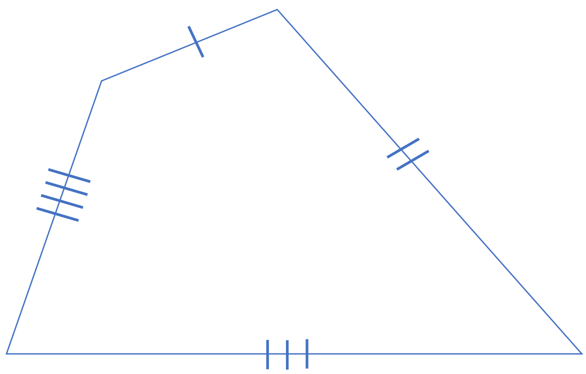
Habits of mind to be developed

- Visualising
- Observing
- Inquiring
- Classifying
- Investigating
- Reasoning
- Comparing
- Justifying

Questions and explanations

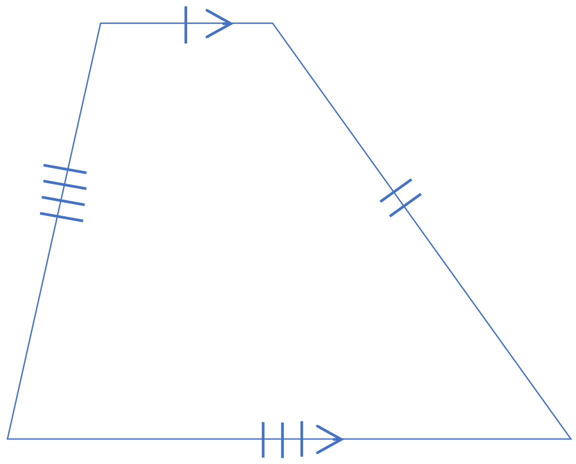
After comparing and discussing the quads below, answer the questions that follow.

Look at how this quad is transformed each time until it becomes a square.



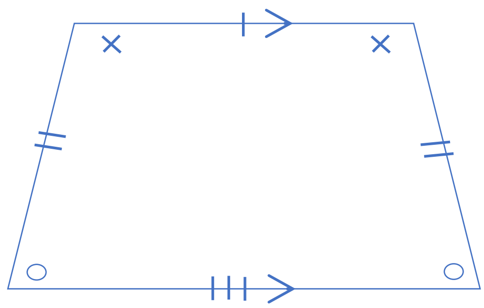
4 sides

____



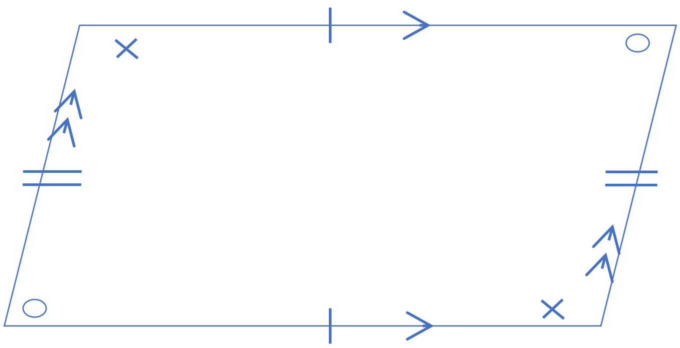
4 sides
1 pair opposite sides //

____



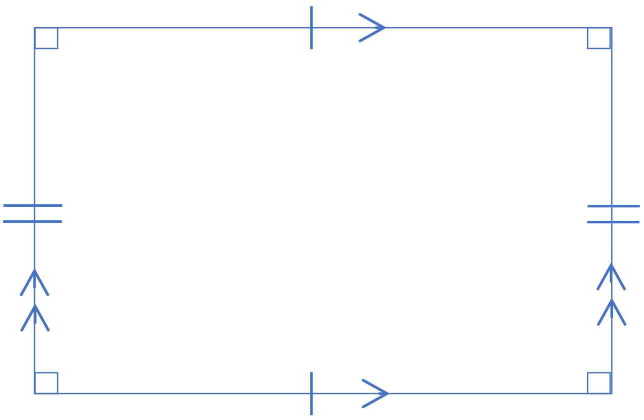
4 sides
1 pair opposite sides //
lateral sides =

____



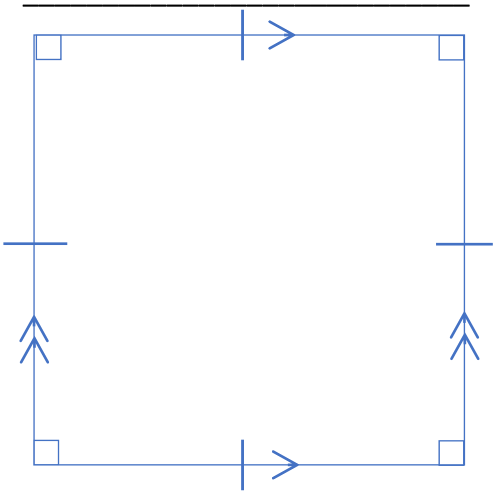
4 sides
opposite sides //
opposite sides =
opposite Ls =

____



4 sides
opposite sides //
opposite sides =
interior Ls = 90°

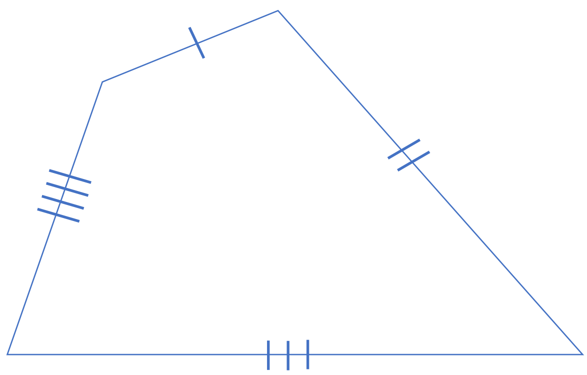
____



4 sides
opposite sides //
all sides =
interior Ls = 90°

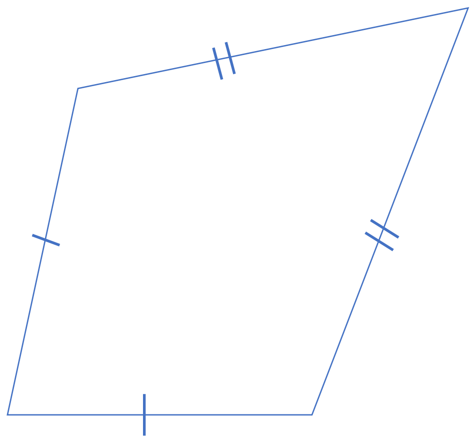
____

Starting again, look how this quad is transformed until it becomes a square.

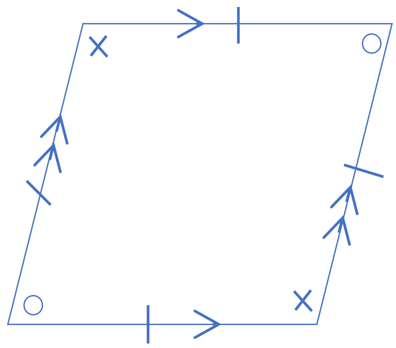


4 sides

____

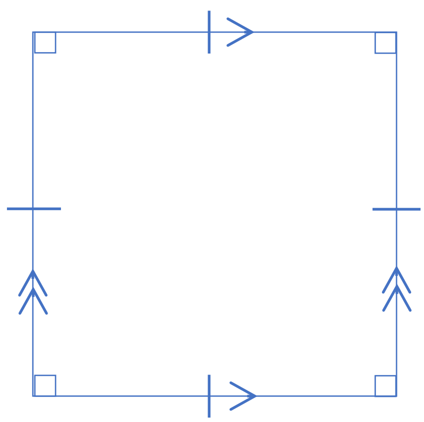


4 sides
2 pairs adjacent sides =
1 pair opposite Ls =



4 sides
opposite sides //
all sides =
opposite Ls =

____



4 sides
opposite sides //
all sides =

interior Ls = 90°

____

Consider the square in terms of inclusivity...

So can we say that all squares are rectangles?
Does a square have:

- 4 sides?
- opposite sides //?
- opposite sides =?
- interior Ls = 90°?

And that all rectangles are parallelograms?
Does a rectangle have:

- 4 sides?
- opposite sides //?
- opposite sides =?
- opposite Ls =?

And that all parallelograms are trapeziums?
Does a parallelogram have:

- 4 sides?
- 1 pair opposite sides //?
- lateral sides =?

Now consider the following:
And are all trapeziums are quadrilaterals?
And are all squares are also rhombi…
And are all rhombi are also kites?
But aren't squares also kites?
Aren't rectangles also quads?

But which rectangles are also squares?

Resources required

No further resources required. A PowerPoint presentation showing the transformation of the quadrilateral must be developed.

Supporting and/or explanatory diagrams and videos

https://www.youtube.com/watch?v=l4S2dX3OLL0

Research articles for support

References for further reading

UNIT 8: Economy of definitions

Content standard which this activity addresses

Economy of definitions

a) Ability to interpret and describe objects and shapes in terms of minimal properties

Intent of this activity

This activity sets out to encourage visualisation and spatial reasoning, and relies on ...

Habits of mind to be developed

- Visualising
- Observing
- Inquiring
- Classifying
- Investigating
- Reasoning
- Comparing
- Justifying

Questions and explanations

Which of the following are definitions of a square?

a) A square is any four-sided shape with all four sides equal in length, and all interior angles 90^{o}
b) A square is any quadrilateral (quad) with all interior angles equal
c) A square is any rectangle with all four sides equal
d) A square is any quad with all sides equal and one interior angle equal to 90^{o}
e) A square is any quad with three sides equal and one interior angle equal to 90^{o}
f) A square is any quad with all sides equal in length
g) A square is any closed shape with all four sides equal in length and opposite angles equal
h) A square is any quad with equal length diagonals which bisect each other at 90^{o}
i) A square is any quad with exactly four lines of symmetry
j) A square is any quad with three sides equal and two interior angles equal to 90^{o}

Resources

No additional resources are needed.

Research articles for support

References for further reading

References

Barclay, N. and Barnes, A., 2013 BIG IDEAS - AN IDEA WITH PRIMARY POTENTIAL? www.atm.org.uk/join/index.html

Niemi, D., Vallone, J., and Vendlinski, T. The Power of Big Ideas in Mathematics Education: Development and Pilot Testing of POWERSOURCE Assessments, CSE Report 697, CRESST/UCL A, August 2006

Izard, V., Pica, P., Spelke, E., and Dehaene, S. PNAS June 14, 2011

Clements, D. H., & Battista, M. T. (1992). Geometry and spatial reasoning. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 420-464). New York: Macmillan.